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On an Extension of Semigroup

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Abstract

This paper contains an ideal extension of semigroup. Suppose that S is a semigroup and Z is a semigroup with zero 0. A semigroup T is called an ideal extension of S with respect to Z if the following conditions are satisfied: (1) S is an ideal of the semigroup T, and (2) Z is Rees quotient semigroup such that Z is isomorphic to T/S. The problem is consist of : (1) what kind of semigroup is T, and (2) what conditions on Z do guarantee the ideal extension. We will start with two extensions of semigroup by using right translation and left translation on semigroup. The object of this paper to present properties of the semigroup T and the conditions on Z.

1 Introduction and Preliminaries

So many attempts have been made for finding the structural properties of left translations and right translations of semigroup. We begin by recalling some results about algebraic semigroup (See [1]).

Let $(S, \cdot)$ be any semigroup with operation $\cdot$, briefly denoted by S . Then, a mapping ψ on S is called a left translation if $\psi(x \cdot y) = \psi(x) \cdot y$ for any elements $x, y \in S$, and a mapping φ on S is called a right translation if $\phi(x \cdot y) = x \cdot \phi(y)$ for any elements $x, y \in S$. A left translation ψ is called inner if there exists an element $s \in S$ such that $\psi(x) = s \cdot x$ for any element $x \in S$, which will be denoted by l_s, that is, $l_s(x) = s \cdot x$. A right translation

ϕ is called inner if there exists an element $s \in S$ such that $\phi(x) = x \cdot s$ for any element $x \in S$, which will be denoted by r_s, that is, $r_s(x) = x \cdot s$.

The set of all left translations on S is the left translation semigroup under the ordinary composition $*$ of mappings. Similarly, the set of all right translations on S is the right translation semigroup under the ordinary composition $\circ$ of mappings. Throughout this paper, $(\Psi(S), *)$ or $\Psi(S)$ will denote the left translation semigroup of S, and $(\Phi(S), \circ)$ or $\Phi(S)$ will denote the right translation semigroup of S.

***Definition** 1.* Let S be an ideal of a semigroup T.
The relation τ defined on T by $x \,\tau\, y$ *if and only if* $x, y \in S$ *or* $x = y$, is a congruence called the *Rees congruence* induced by S.
The quotient semigroup T/τ is the *Rees quotient semigroup* relative to S and is denoted by T/S

***Definition** 2.* Let S be a semigroup and Z be a semigroup with zero 0.
(1) Semigroup $\mathcal{E}_L$ is called a left ideal extension of S by Z if S is a left ideal of $\mathcal{E}_L$ and Z is isomorphic to the Rees quotient semigroup $\mathcal{E}_L/S$.
(2) Semigroup $\mathcal{E}_R$ is called a right ideal extension of S by Z if S is a right ideal of $\mathcal{E}_R$ and Z is isomorphic to the Rees quotient semigroup $\mathcal{E}_R/S$.

***Definition** 3.* Given a semigroup S and a semigroup Z with zero 0, semigroup T is called an ideal extension of S by Z if S is a two-sided ideal of T and there exists an isomorphism from Z to Rees quotient semigroup T/S.

Now we can define two semigroups, $(\mathcal{T}_L, \circledast) = (S, \cdot) \cup (\Psi(S), *)$ and $(\mathcal{T}_R, \circledcirc) = (S, \cdot) \cup (\Phi(S), \circ)$, with semigroup S as right and left ideals of $\mathcal{T}_L$ and $\mathcal{T}_R$, respectively.

1. Let x, y be any elements of $(S, \cdot)$ and ψ, φ be any elements of $(\Psi(S), *)$. The operation $\circledast$ on $\mathcal{T}_L$ will be defined by
   $(1) x \circledast y = x \cdot y$, $(2) x \circledast \psi = l_x * \psi$, $(3) \psi \circledast x = \psi(x)$ and $(4) \psi \circledast \varphi = \psi * \varphi$, where $\psi * \varphi$ is defined by $\psi * \varphi(z) = \psi(\varphi(z))$ and $l_x(z) = z \cdot x$ for all $z \in S$.

2. Let x, y be any elements of $(S, \cdot)$ and ϕ, v be any elements of $(\Phi(S), \circ)$. The operation $\circledcirc$ on $\mathcal{T}_R$ will be defined by
   $(1) x \circledcirc y = x \cdot y$, $(2) x \circledcirc \phi = \phi(x)$, $(3) \phi \circledcirc x = \phi \circ r_x$ and $(4) \phi \circledcirc v = \phi \circ v$, where $\phi \circ v$ is defined by $\phi \circ v(z) = v(\phi(z))$ and $r_x = x \cdot z$ for all $z \in S$.

To show that $\mathcal{T}_L$ is a semigroup, it is necessary that the following eight equations hold for all $x, y, z \in S$ and $\psi, \varphi, \chi \in \Psi(S)$:

$$(1)(x \circledast y) \circledast z = x \circledast (y \circledast z), \qquad (5)(\psi \circledast \varphi) \circledast \chi = \psi \circledast (\varphi \circledast \chi),$$
$$(2)(x \circledast \psi) \circledast \varphi = x \circledast (\psi \circledast \varphi), \qquad (6)(\psi \circledast x) \circledast y = \psi \circledast (x \circledast y),$$
$$(3)(x \circledast \psi) \circledast y = x \circledast (\psi \circledast y), \qquad (7)(\psi \circledast \varphi) \circledast x = \psi \circledast (\varphi \circledast x),$$
$$(4)(x \circledast y) \circledast \varphi = x \circledast (y \circledast \varphi) \text{ and } \quad (8)(\psi \circledast x) \circledast \varphi = \psi \circledast (x \circledast \varphi)$$

It is shown that equations (1) to (8) hold on $\mathcal{T}_L$ as follows:

(1) $(x \circledast y) \circledast z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \circledast (y \circledast z)$;

(2) $(x \circledast \psi) \circledast \varphi = (l_x * \psi) * \varphi = l_x * (\psi * \varphi) = x \circledast (\psi \circledast \varphi)$;

(3) $(x \circledast \psi) \circledast y = (l_x * \psi) \circledast y = (l_x * \psi)(y) = l_x(\psi(y)) = l_x(\psi \circledast y)$;

(4) $(x \circledast y) \circledast \varphi = (x \cdot y) \circledast \varphi = l_{x \cdot y} * \varphi$. Since $l_{x \cdot y}(z) = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \cdot l_y(z) = l_x(l_y(z)) = l_x * l_y(z)$ for any $z \in S$, it is obtained that $l_{x \cdot y} * \varphi = (l_x * l_y) * \varphi = l_x * (l_y * \varphi) = x \circledast (l_y * \varphi) = x \circledast (y \circledast \varphi)$;

(5) $(\psi \circledast \varphi) \circledast \chi = (\psi * \varphi) * \chi = \psi * (\varphi * \chi) = \psi \circledast (\varphi \circledast \chi)$;

(6) $(\psi \circledast x) \circledast y = \psi(x) \circledast y = \psi(x) \cdot y = \psi(x \cdot y)$ and $\psi \circledast (x \circledast y) = \psi \circledast (x \cdot y) = \psi(x \cdot y)$;

(7) $(\psi \circledast \varphi) \circledast x = (\psi * \varphi) \circledast x = \psi * \varphi(x) = \psi(\varphi(x)) = \psi(\varphi \circledast x) = \psi \circledast (\varphi \circledast x)$;

(8) $(\psi \circledast x) \circledast \varphi = (\psi(x)) \circledast \varphi = l_{\psi(x)} * \varphi$ and $\psi \circledast (x \circledast \varphi) = \psi * (l_x * \varphi)$. From the fact that for any $z \in S$, $(l_{\psi(x)} * \varphi)(z) = l_{\psi(x)}(\varphi(z)) = \psi(x) \cdot \varphi(z)$ and $\psi * (l_x * \varphi)(z) = \psi((l_x * \varphi)(z)) = \psi((l_x(\varphi(z))) = \psi(x \cdot \varphi(z)) = \psi(x) \cdot \varphi(z)$, we have $l_{\psi(x)} * \varphi = \psi * (l_x * \varphi)$.

It is also shown that $\mathcal{T}_R$ is a semigroup in the same manner, that is, the following eight equations hold for all $x, y, z \in S$ and $\phi, \upsilon, \omega \in \Phi(S)$.

$$(1)(x \circledcirc y) \circledcirc z = x \circledcirc (y \circledcirc z) \qquad (5)(\phi \circledcirc \upsilon) \circledcirc \omega = \phi \circledcirc (\upsilon \circledcirc \omega)$$
$$(2)(x \circledcirc \phi) \circledcirc \upsilon = x \circledcirc (\phi \circledcirc \upsilon) \qquad (6)(\phi \circledcirc x) \circledcirc y = \phi \circledcirc (x \circledcirc y)$$
$$(3)(x \circledcirc \phi) \circledcirc y = x \circledcirc (\phi \circledcirc y) \qquad (7)(\phi \circledcirc \upsilon) \circledcirc x = \phi \circledcirc (\upsilon \circledcirc x)$$
$$(4)(x \circledcirc y) \circledcirc \upsilon = x \circledcirc (y \circledcirc \upsilon) \qquad (8)(\phi \circledcirc x) \circledcirc \upsilon = \phi \circledcirc (x \circledcirc \upsilon)$$

In fact, it is be shown that equations (1) to (8) hold on $\mathcal{T}_R$ as follows:

(1) $(x \circledcirc y) \circledcirc z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \circledcirc (y \circledcirc z)$;

(2) $(x \circledcirc \phi) \circledcirc v = \phi(x) \circledcirc v = v(\phi(x)) = \phi \circ v(x) = x \circledcirc (\phi \circledcirc v)$;

(3) $(x \circledcirc \phi) \circledcirc y = \phi(x) \circledcirc y = \phi(x) \cdot y$ and $x \circledcirc (\phi \circledcirc y) = x \circledcirc (\phi \circ r_y) = r_y(\phi(x)) = \phi(x) \cdot y$;

(4) $(x \circledcirc y) \circledcirc v = (x \cdot y) \circledcirc v = v(x \cdot y) = x \cdot v(y)$ and $x \circledcirc (y \circledcirc v) = x \circledcirc (y \circledcirc v)$

(5) $(\phi \circledcirc v) \circledcirc \omega = (\phi \circ v) \circ \omega = \phi \circ (v \circ \omega) \doteq \phi \circledcirc (v \circledcirc \omega)$;

(6) $(\phi \circledcirc x) \circledcirc y = (\phi \circ r_x) \circledcirc y = (\phi \circ r_x) \circ r_y = \phi \circ r_x \circ r_y$ and $\phi \circledcirc (x \circledcirc y) = \phi \circledcirc (x \cdot y) = \phi \circ r_{x \cdot y}$. Since $r_{x \cdot y}(z) = z \cdot (x \cdot y) = (z \cdot x) \cdot y = (r_x(z)) \cdot y = r_y(r_x(z)) = r_x \circ r_y(z)$ for any $z \in S$, it is obtained that $\phi \circ r_{x \cdot y} = \phi \circ r_x \circ r_y$;

(7) $(\phi \circledcirc v) \circledcirc x = (\phi \circ v) \circledcirc x = (\phi \circ v) \circ r_x = \phi \circ (v \circ r_x) = \phi \circ (v \circledcirc x) = \phi \circledcirc (v \circledcirc x)$;

(8) $(\phi \circledcirc x) \circledcirc v = (\phi \circ r_x) \circledcirc v = \phi \circ r_x \circ v$ and $\phi \circledcirc (x \circledcirc v) = \phi \circledcirc v(x) = \phi \circ r_{v(x)}$. From the fact that for any $z \in S$, $((\phi \circ r_x) \circ v)(z) = v(\phi \circ r_x(z)) = v(r_x(\phi(z))) = v(\phi(z) \cdot x) = \phi(z) \cdot v(x)$ and $(\phi \circ r_{v(x)})(z) = r_{v(x)}(\phi(z)) = \phi(z) \cdot v(x)$, we have $(\phi \circ r_x) \circ v = (\phi \circ r_x) \circ v = \phi \circ (r_x \circ v)$.

Definition 4. Let S be a semigroup and Z be a set such that the union of S and Z, $T = S \cup Z$, is a semigroup.
(1) The semigroup $(T, \cdot)$ is called a quasi-inflation of S with respect to Z if there exists a homomorphism f from T onto S such that $f(x) = x$ for all $x \in S$.
(2) The semigroup $(T, \cdot)$ is called an inflation of $(S, *)$ with respect to Z if there exists a homomorphism f from T onto S such that $f(x) = x$ for all $x \in S$ and $x \cdot y = f(x) * f(y)$ for all $x, y \in T$.

2 Ideal Extensions

Lemma 1. Let S be a subsemigroup of T.
(1) If S is a left ideal of T, then T is isomorphich to a subsemigroup of $\Psi(S)$.
(2) If S is a left ideal of T, then T is isomorphic to a subsemigroup of $\Phi(S)$.

[*Proof*] From the definitions of $\Psi(S)$ and left ideal, for any element s of T, there exists a transformation f_s on S, defined by $f_s(x) = s \cdot x$ for all $x \in S$. Since $f_s(x \cdot y) = s \cdot x \cdot y = f_s(x) \cdot y$, it is obvious that f_s is a left translation. Similarly from the definitions of $\Phi(S)$ and right ideal, the transformation g_t on S, defined by $g_t(x) = x \cdot t$ for all $s \in S$, is a left translation.

From above lemma, we have the following proposition:

***Proposition** 1.* Let S be a subsemigroup of T.
(1) There exists is a left ideal extension of S by Z if and only if $Z \setminus \{0\}$ is embedded in a quai-inflation of $\Psi(S)$.
(2) There exists is a right ideal extension of S by Z if and only if $Z \setminus \{0\}$ is embedded in a quai-inflation of $\Phi(S)$.

Let now ϕ and ψ be a right and a left translations of a semigroup $(S, \cdot)$, respectively. Then ϕ and ψ are called linked if $x \cdot \psi(y) = \phi(x) \cdot y$ for all $x, y \in S$ and denoted by $\phi \sim \psi$. The set $\mathcal{LK}(S)$ is defined by $\mathcal{LK}(S) = \{[\phi, \psi] : \phi \sim \psi, \phi \in \Phi(S), \psi \in \Psi(S)\}$, that is, $\mathcal{LK}(S)$ is a semigroup called translation hull with the operation $\triangle$ defined by $[\phi, \psi] \triangle [\phi', \psi'] = [\phi \circ \phi', \psi * \psi']$.

***Definition** 5.* Let S be a semigroup and $\mathcal{LK}(S)$ be the translation hull of S and suppose that $[\phi, \psi]$ and $[\phi', \psi']$ are elements of $\mathcal{LK}(S)$. $[\phi, \psi]$ is commutable to $[\phi', \psi']$ if ψ and ϕ' are commutative, i.e., $\psi(\phi'(x)) = \phi'(\psi(x))$, which is denoted by $[\phi, \psi] \ltimes [\phi', \psi']$. A subsemigroup $\mathcal{H}$ of $\mathcal{LK}(S)$ is called commutable if $[\phi, \psi] \ltimes [\phi', \psi']$ for any pairs $[\phi, \psi]$ and $[\phi', \psi'] \in \mathcal{H}$.

Now we can define a semigroup, $T(S : \mathcal{H}) = S \cup \mathcal{H}$, where $\mathcal{H}$ is a commutable subsemigroup of $\mathcal{LK}(S)$.
In fact, let x, y, z be any elements of S and $[\phi_1, \psi_1], [\phi_2, \psi_2]$ be any elements of $\mathcal{LK}(S)$, then an operation $\boxtimes$ on $T(S : \mathcal{H})$ can be defined as follows:

(1) $x \boxtimes y = x \cdot y$;

(2) $x \boxtimes [\phi_1, \psi_1] = \phi_1(x)$;

(3) $[\phi_1, \psi_1] \boxtimes x = \psi_1(x)$;

(4) $[\phi_1, \psi_1] \boxtimes [\phi_2, \psi_2]; = [\phi_1, \psi_1] \triangle [\phi_2, \psi_2]$.

Lemma 2. Let S be a subsemigroup of T.
If S is an ideal of T, then T is isomorphic to a commutable subsemigroup of the translation hull $\mathcal{LK}(S)$.

[*Proof*] From the definitions of $\mathcal{LK}(S)$ and ideal, for any elements a and b of T, there exists an element $[\phi_a, \psi_b]$ of $\mathcal{LK}(S)$, which is a pair of a right translation ϕ_a and a left translation ψ_b on S such that $x \boxtimes [\phi_a, \psi_b] = \phi_a(x) = x \cdot a$ and $[\phi_a, \psi_b] \boxtimes x = \psi_b(x) = b \cdot x$ for all $x \in S$. Since $\psi_b(\phi_a(x)) = \psi_b(x \cdot a) = b \cdot (x \cdot a) = (b \cdot x) \cdot a = \phi_a(\psi_b(x))$ for all $x \in S$, it is obvious that $\{[\phi_a, \psi_b] : a, b \in T\}$ is commutable.

$T(S : \mathcal{H})$ is a semigroup because of obtaining the following eight equations which hold for all $x, y, z \in S$ and $[\phi_1, \psi_1], [\phi_2, \psi_2], [\phi_3, \psi_3] \in T(S : \mathcal{H})$.

(1) $(x \boxtimes y) \boxtimes z = x \boxtimes (y \boxtimes z)$;

(2) $(x \boxtimes [\phi_1, \psi_1]) \boxtimes [\phi_2, \psi_2] = x \boxtimes ([\phi_1, \psi_1] \boxtimes [\phi_2, \psi_2])$;

(3) $(x \boxtimes [\phi_1, \psi_1]) \boxtimes y = x \boxtimes ([\phi_1, \psi_1] \boxtimes y)$;

(4) $(x \boxtimes y) \boxtimes [\phi_1, \psi_1] = x \boxtimes (y \boxtimes [\phi_1, \psi_1])$;

(5) $([\phi_1, \psi_1] \boxtimes [\phi_2, \psi_2]) \boxtimes [\phi_3, \psi_3] = [\phi_1, \psi_1] \boxtimes ([\phi_2, \psi_2] \boxtimes [\phi_3, \psi_3])$

(6) $([\phi_1, \psi_1] \boxtimes x) \boxtimes y = [\phi_1, \psi_1] \boxtimes (x \boxtimes y)$;

(7) $([\phi_1, \psi_1] \boxtimes [\phi_2, \psi_2]) \boxtimes x = [\phi_1, \psi_1] \boxtimes ([\phi_2, \psi_2] \boxtimes x)$;

(8) $([\phi_1, \psi_1] \boxtimes x) \boxtimes [\phi_2, \psi_2] = [\phi_1, \psi_1] \boxtimes (x \boxtimes [\phi_2, \psi_2])$.

It is shown that equations (1) to (8) hold on $T(S : H)$ as follows:

(1) $(x \boxtimes y) \boxtimes z = (x \cdot y) \cdot z = x \cdot (y \cdot z) = x \boxtimes (y \boxtimes z)$;

(2) $(x \boxtimes [\phi_1, \psi_1]) \boxtimes [\phi_2, \psi_2] = \phi_1(x) \boxtimes [\phi_2, \psi_2] = \phi_2(\phi_1(x)) = (\phi_1 \circ \phi_2)(x) = x \boxtimes ([\phi_1 \circ \phi_2, \psi_1 * \psi_2]) = x \boxtimes ([\phi_1, \psi_1] \boxtimes [\phi_2, \psi_2])$;

(3) $(x \boxtimes [\phi_1, \psi_1]) \boxtimes y = \phi_1(x) \boxtimes y = \phi_1(x) \cdot y = x \cdot \psi_1(y) = x \boxtimes \psi_1(y) = x \boxtimes ([\phi_1, \psi_1] \boxtimes y)$ because of $\phi_1 \sim \psi_1$;

(4) $(x \boxtimes y) \boxtimes [\phi_1, \psi_1] = (x \cdot y) \boxtimes [\phi_1, \psi_1] = \phi_1(x \cdot y) = x \cdot \phi_1(y) = x \boxtimes (y \boxtimes [\phi_1, \psi_1])$;

(5) $([\phi_1,\psi_1] \boxtimes [\phi_2,\psi_2]) \boxtimes [\phi_3,\psi_3] = ([\phi_1,\psi_1] \vartriangle [\phi_2,\psi_2]) \vartriangle [\phi_3,\psi_3] = [(\phi_1 \circ \phi_2) \circ \phi_3, (\psi_1 * \psi_2) * \psi_3] = [\phi_1 \circ (\phi_2 \circ \phi_3), \psi_1 * (\psi_2 * \psi_3)] = [\phi_1,\psi_1] \vartriangle ([\phi_2,\psi_2] \vartriangle [\phi_3,\psi_3]) = [\phi_1,\psi_1] \boxtimes ([\phi_2,\psi_2] \boxtimes [\phi_3,\psi_3])$;

(6) $([\phi_1,\psi_1] \boxtimes x) \boxtimes y = \psi_1(x) \boxtimes y = \psi_1(x) \cdot y = \psi_1(x \cdot y) = [\phi_1,\psi_1] \boxtimes (x \cdot y) = [\phi_1,\psi_1] \boxtimes (x \boxtimes y)$;

(7) $([\phi_1,\psi_1] \boxtimes [\phi_2,\psi_2]) \boxtimes x = ([\phi_1 \circ \phi_2, \psi_1 * \psi_2]) \boxtimes x = ([\phi_1 \circ \phi_2, \psi_1 * \psi_2])(x) = \psi_1 * \psi_2(x) = \psi_1(\psi_2(x)) = \psi_1([\phi_2,\psi_2] \boxtimes x) = [\phi_1,\psi_1] \boxtimes ([\phi_2,\psi_2] \boxtimes x)$;

(8) $([\phi_1,\psi_1] \boxtimes x) \boxtimes [\phi_2,\psi_2] = \psi_1(x) \boxtimes [\phi_2,\psi_2] = \phi_2(\psi_1(x)) = \psi_1(\phi_2(x)) = \psi_1(x \boxtimes [\phi_2,\psi_2]) = [\phi_1,\psi_1] \boxtimes (x \boxtimes [\phi_2,\psi_2])$ because of $[\phi_1,\psi_1] \ltimes [\phi_2,\psi_2]$.

From above lemma, we have the following proposition:

***Proposition** 2.* There exists is an ideal extension of S by Z if and only if $Z \setminus \{0\}$ is embedded in a quai-inflation of a commutable subsemigroup of translation hull tion of $\mathcal{LK}(S)$.

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